MODAL INFORMATION LOGIC: DECIDABILITY AND COMPLETENESS

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Extract of MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili
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Plan for the talk

- Introducing the logics
- Stating the problems
- Outlining the strategy
- Solving the problems using the strategy
Defining (the basic) modal information logics (MILs)

Definition (language and semantics)
The language is given by

\[ \varphi ::= \bot \mid p \mid \neg \varphi \mid \varphi \lor \psi \mid \langle \text{sup} \rangle \varphi \psi, \]

and the semantics of ‘\langle \text{sup} \rangle’ is:

\[ w \vDash \langle \text{sup} \rangle \varphi \psi \iff \exists u, v (u \vDash \varphi; v \vDash \psi; \ w = \text{sup}\{u, v\}) \]

Example

\[ w \vDash \langle \text{sup} \rangle pq \]

\[ u \vDash p \]

\[ v \vDash q \]

Definition (frames and logics)
Three classes of frames \((W, \leq)\), namely those where

(Pre) \((W, \leq)\) is a preorder (refl., tr.);

(Pos) \((W, \leq)\) is a poset (anti-sym. preorder); and

(Sem) \((W, \leq)\) is a join-semilattice (poset w. all bin. joins)

Resulting in the logics \(\text{MIL}_{\text{Pre}}, \text{MIL}_{\text{Pos}}, \text{MIL}_{\text{Sem}}\), respectively.
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\[ w \models \langle \text{sup} \rangle pq \]

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\[ \begin{align*}
(\text{Pre}) \quad & (W, \leq) \text{ is a preorder (refl., tr.);} \\
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(\text{Sem}) \quad & (W, \leq) \text{ is a join-semilattice (poset w. all bin. joins)}
\end{align*} \]

Resulting in the logics \(\text{MIL}_{\text{Pre}}, \text{MIL}_{\text{Pos}}, \text{MIL}_{\text{Sem}}, \) respectively.
Defining (the basic) modal information logics (MILs)

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Resulting in the *logics* \(\text{MIL}_{Pre}, \text{MIL}_{Pos}, \text{MIL}_{Sem}\), respectively.
Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker logics).
- Introduced to model a theory of information (by van Bentham (1996)).
- Modestly extend $\mathbf{S4}$ [$MIL_{\text{Pre}}, MIL_{\text{Pos}}$].

What in particular?

Guided by two central problems (posed in van Bentham (2017, 2019)), namely

(A) axiomatizing $MIL_{\text{Pre}}$ and $MIL_{\text{Pos}}$; and

(D) proving (un)decidability.
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Initial study \((MIL_{Pre} \text{ and } MIL_{Pos})\)

**Proposition**

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

1. We axiomatize \(MIL_{Pre}\) (and deduce \(MIL_{Pre} = MIL_{Pos}\)).
2. Use the axiomatization to find another class of structures \(C\) for which \(\text{Log}(C) = MIL_{Pre}\).
3. Prove that on \(C\) we do have the FMP and deduce decidability.
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Axiomatization (soundness and completeness)

\( \text{MIL}_{\text{Pre}} \) is (sound and complete w.r.t.) the least normal modal logic with axioms:

- (Re.) \( p \land q \rightarrow \langle \text{sup} \rangle pq \)
- (4) \( PPp \rightarrow Pp \)
- (Co.) \( \langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp \)
- (Dk.) \( (p \land \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq \)

Proof idea

Soundness

For completeness, let \( \Gamma \supseteq \Gamma_0 \) be an MCS extending some consistent \( \Gamma_0 \). We construct a satisfying model using the step-by-step method:

- (Base) Singleton frame \( F_0 := (\{x_0\}, \{(x_0, x_0)\}) \) and ‘labeling’ \( l_0(x_0) = \Gamma \).
- (Ind) Suppose \((F_n, l_n)\) has been constructed.
  - If \( x \in F_n \) and \( \neg \langle \text{sup} \rangle \psi \psi' \in l_n(x) \) but \( x = \text{sup}_n \{y, z\} \) s.t. \( \psi \in l_n(y), \psi' \in l_n(z) \), coherently extend to \((F_{n+1}, l_{n+1}) \supseteq (F_n, l_n) \) so that \( x \neq \text{sup}_{n+1} \{y, z\} \).
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(1): axiomatizing $\text{MIL}_{\text{Pre}}$

**Axiomatization (soundness and completeness)**

$\text{MIL}_{\text{Pre}}$ is (sound and complete w.r.t.) the least normal modal logic with axioms:

- **(Re.)** $p \wedge q \rightarrow \langle \sup \rangle pq$
- **(4)** $PPp \rightarrow Pp$
- **(Co.)** $\langle \sup \rangle pq \rightarrow \langle \sup \rangle qp$
- **(Dk.)** $(p \wedge \langle \sup \rangle qr) \rightarrow \langle \sup \rangle pq$

**Proof idea**

**Soundness ✔**

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent $\Gamma_0$. We construct a satisfying model using the step-by-step method:

- **(Base)** Singleton frame $F_0 := (\{x_0\}, \{(x_0, x_0)\})$ and ‘labeling’ $l_0(x_0) = \Gamma$.

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  - If $x \in F_n$ and $\neg \langle \sup \rangle \psi \psi' \in l_n(x)$ but $x = \sup_n \{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, coherently extend to $(F_{n+1}, l_{n+1}) \supseteq (F_n, l_n)$ so that $x \neq \sup_{n+1} \{y, z\}$.
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- Similarly, for $\langle \sup \rangle \chi \chi' \in l_n(x)$.
Completeness of $MIL_{Pre}$ (cont.)

Example

\[
\{\langle \sup \rangle \chi_0 \chi_0', \langle \sup \rangle \chi_1 \chi_1' \} \subseteq l(x)
\]

\[
\langle \sup \rangle \text{-repair} \leadsto \langle \sup \rangle \text{-repair} \leadsto \neg \langle \sup \rangle \text{-repair}
\]

\[
x \in l(y) \quad x_0 \in l(z)
\]

\[
\neg \langle \sup \rangle \psi \psi' \in l(x)
\]

\[
\neg \langle \sup \rangle \text{-repair}
\]

\[
y \quad \psi \in l(z)
\]

\[
z \quad \psi' \in l(z')
\]

\[
x \quad d
\]

\[
y' \quad \psi' \in l(y')
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(1): axiomatizing $\text{MIL}_{\text{Pre}}$

**Axiomatization (soundness and completeness)**

$\text{MIL}_{\text{Pre}}$ is (sound and complete w.r.t.) the least normal modal logic with axioms:

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- **(Dk.)** $(p \land \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

**About the proof**

Soundness: routine.

**Corollary**

As a corollary we get that $\text{MIL}_{\text{Pre}} = \text{MIL}_{\text{Pos}}$. 
Axiomatization (soundness and completeness)

\( \text{MIL}_{\text{Pre}} \) is (sound and complete w.r.t.) the least normal modal logic with axioms:

\begin{align*}
(\text{Re.}) & \quad p \land q \rightarrow \langle \text{sup} \rangle pq \\
(4) & \quad P P p \rightarrow P p \\
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(\text{Dk.}) & \quad (p \land \langle \text{sup} \rangle q r) \rightarrow \langle \text{sup} \rangle p q
\end{align*}

About the proof

Soundness: routine.

Corollary

As a corollary we get that \( \text{MIL}_{\text{Pre}} = \text{MIL}_{\text{Pos}} \).
(2) and (3): ‘decidability via completeness’

(2) Find another class $C$ for which $\text{Log}(C) = \text{MIL}_{\text{Pre}}$:

(i) Nothing in the ax. of $\text{MIL}_{\text{Pre}}$ necessitating ‘$\langle \sup \rangle$’ to be interpreted using a supremum relation.

(ii) Canon. re-interpretation:

$$C := \{(W, C) \mid (W, C) \Vdash (\text{Re.}) \land (\text{Co.}) \land (4) \land (\text{Dk.})\},$$

where $C' \subseteq W^3$ is an arbitrary relation.

(iii) Then $\text{Log}(C) = \text{MIL}_{\text{Pre}}$.

(3) Decidability through FMP on $C$:

(i) On $C$, we get the FMP through filtration.

(ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.
(2) and (3): ‘decidability via completeness’

(2) Find another class $\mathcal{C}$ for which $\text{Log}(\mathcal{C}) = \text{MIL}_{\text{Pre}}$:
   (i) Nothing in the ax. of $\text{MIL}_{\text{Pre}}$ necessitating ‘$\langle \sup \rangle$’ to be interpreted using a supremum relation.
   (ii) Canon. re-interpretation:

   \[
   \mathcal{C} := \left\{ (W, C) \mid (W, C) \models (\text{Re.}) \land (\text{Co.}) \land (4) \land (\text{Dk.}) \right\},
   \]

   where $C \subseteq W^3$ is an arbitrary relation.
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Gen. takeaway: When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.
(2) Find another class \( C \) for which \( \log(C) = \text{MIL}_{\text{Pre}} \):

(i) Nothing in the ax. of \( \text{MIL}_{\text{Pre}} \) necessitating ‘\( \langle \sup \rangle \)’ to be interpreted using a suprimum relation.

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C := \{(W, C) \mid (W, C) \vDash (\text{Re.}) \land (\text{Co.}) \land (4) \land (\text{Dk.})\},
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(2) Find another class $C$ for which $\text{Log}(C) = \text{MIL}_{\text{Pre}}$:

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(2) Find another class \( C \) for which \( \log(\mathcal{C}) = \text{MIL}_{\text{Pre}} \):
   
   (i) Nothing in the ax. of \( \text{MIL}_{\text{Pre}} \) necessitating ‘\( \langle \sup \rangle \)’ to be interpreted using a \text{supremum} relation.
   
   (ii) Canon. re-interpretation:

   \[
   \mathcal{C} := \{(W, C) \mid (W, C) \models (R_e.) \land (C_0.) \land (4) \land (Dk.)\},
   \]

   where \( C \subseteq W^3 \) is an \text{arbitrary} relation.
   
   (iii) Then \( \log(\mathcal{C}) = \text{MIL}_{\text{Pre}} \).

(3) **Decidability through FMP on \( \mathcal{C} \):**

   (i) On \( \mathcal{C} \), we get the FMP through filtration.
   
   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

**Gen. takeaway:** When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.
(2) and (3): ‘decidability via completeness’

(2) Find another class \( \mathcal{C} \) for which \( \text{Log}(\mathcal{C}) = \text{MIL}_{\text{Pre}} \):

(i) Nothing in the ax. of \( \text{MIL}_{\text{Pre}} \) necessitating \( \langle \sup \rangle \) to be interpreted using a \textit{supremum} relation.
(ii) Canon. re-interpretation:

\[
\mathcal{C} := \{(W, C) \mid (W, C) \models (\text{Re.}) \land (\text{Co.}) \land (4) \land (Dk.)\},
\]

where \( C \subseteq W^3 \) is an \textit{arbitrary} relation.
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(2) Find another class \( C \) for which \( \text{Log}(C) = \text{MIL}_{\text{Pre}} \):

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C := \{(W, C') \mid (W, C') \models (\text{Re.}) \land (\text{Co.}) \land (4) \land (Dk.)\},
\]

where \( C' \subseteq W^3 \) is an arbitrary relation.

(iii) Then \( \text{Log}(C) = \text{MIL}_{\text{Pre}} \).

(3) Decidability through FMP on \( C \):

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Gen. takeaway: When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.
(2) and (3): ‘decidability via completeness’

(2) Find another class $C$ for which $\text{Log}(C) = \text{MIL}_{\text{Pre}}$:
   (i) Nothing in the ax. of $\text{MIL}_{\text{Pre}}$ necessitating ‘$\langle \sup \rangle$’ to be interpreted using a suprenum relation.
   (ii) Canon. re-interpretation:
   $$C := \{ (W, C') \mid (W, C') \vdash (\text{Re.}) \land (\text{Co.}) \land (4) \land (Dk.) \},$$
   where $C' \subseteq W^3$ is an arbitrary relation.
   (iii) Then $\text{Log}(C) = \text{MIL}_{\text{Pre}}$.

(3) Decidability through FMP on $C$:
   (i) On $C$, we get the FMP through filtration.
   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

**Gen. takeaway:** When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.
How about join-semilattices (i.e., $\text{MIL}_{\text{Sem}}$)?
Axiomatizing \( \text{MIL}_{\text{Sem}} \)

Three ways to completeness (some intuitions for our proof):

**Henkin (e.g., K)**

\[ M \]

**Standard step-by-step (e.g., MIL\(_{\text{Pre}}\))**

\[ M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_\omega \]

**‘Indeterministic step-by-step’ (MIL\(_{\text{Sem}}\))**

Model constr.: 

\[ M_0 \rightarrow M_{01} \rightarrow M_{011} \rightarrow \cdots \]

Axioms: 

\[ \pi_0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \cdots \]
Axiomatizing $\text{MIL}_{Sem}$

Three ways to completeness (some intuitions for our proof):

**Henkin (e.g., $K$)**

- Model constr.
- Axioms: $\pi_0$, $\pi_1$, $\pi_2$, ...$

**Standard step-by-step (e.g., $\text{MIL}_{Pre}$)**

- $M_0$, $M_1$, $M_2$, ..., $M_\infty$

**‘Indeterministic step-by-step’ ($\text{MIL}_{Sem}$)**

- Model constr.
- Axioms: $\pi_0$, $\pi_1$, $\pi_2$, ...$

\pi_0$ $\pi_1$ $\pi_2$
Axiomatizing $\textit{MIL}_{Sem}$

Three ways to completeness (some intuitions for our proof):

**Henkin (e.g., K)**

$M_0$

**Standard step-by-step (e.g., $\textit{MIL}_{Pre}$)**

$M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_\omega$

‘Indeterministic step-by-step’ ($\textit{MIL}_{Sem}$)

Model constr.: \[ M_0 \rightarrow M_{01} \rightarrow \cdots \rightarrow M_{01n_01} \rightarrow \cdots \]

Axioms: $\pi_0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \cdots$
### Axiomatizing $\text{MIL}_{\text{Sem}}$

#### Three ways to completeness (some intuitions for our proof):

<table>
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<th>Standard step-by-step (e.g., $\text{MIL}_{\text{Pre}}$)</th>
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<tbody>
<tr>
<td>$M_0$</td>
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#### ‘Indeterministic step-by-step’ ($\text{MIL}_{\text{Sem}}$)

**Model constr.:**

- $M_0$
- $M_{01}$
- $M_{011}$
- $\cdots$
- $M_{0n0}$
- $M_{0n0n0}$
- $\cdots$

**Axioms:**

- $\pi_0$
- $\pi_1$
- $\pi_2$
- $\cdots$
Thank you!


Can we generalize these techniques?
(Natural) extensions of $MIL_{Pre}$ and $MIL_{Pos}$ [and $S4$] are obtained by adding an informational implication ‘\’.

**Definition**

The language is given by adding ‘\’ with semantics:

\[
v \vDash \varphi \backslash \psi \quad \mathrm{iff} \quad \forall u, w ([u \vDash \varphi, w = \sup\{u, v\}] \Rightarrow w \vDash \psi)
\]

We denote the resulting logics as $MIL_{\text{Pre}}$, $MIL_{\text{Pos}}$, respectively.

The problems now become

(A) axiomatizing $MIL_{\text{Pre}}$ and $MIL_{\text{Pos}}$; and
(D) proving (un)decidability.

The same (1)-(2)-(3) structure is used as before, but now we

(1’) axiomatize the logic $\text{Log}_{\cdot}(\mathcal{C})$;
(2’) through representation show that $\text{Log}_{\cdot}(\mathcal{C}) = MIL_{\text{Pre}} = MIL_{\text{Pos}}$; and
(3) get decidability through FMP on $\mathcal{C}$.
(Natural) extensions of $\text{MIL}_{\text{Pre}}$ and $\text{MIL}_{\text{Pos}}$ [and $\textbf{S4}$] are obtained by adding an informational implication ‘\’.

**Definition**

The language is given by adding ‘\’ with semantics:

$$v \models \varphi \setminus \psi \iff \forall u, w ([u \models \varphi, w = \sup\{u, v\}] \Rightarrow w \models \psi)$$

We denote the resulting logics as $\text{MIL}_{\text{-Pre}}$, $\text{MIL}_{\text{-Pos}}$, respectively.

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(A) aximatizing $\text{MIL}_{\text{-Pre}}$ and $\text{MIL}_{\text{-Pos}}$; and
(D) proving (un)decidability.

The same (1)-(2)-(3) structure is used as before, but now we

(1’) aximatize the logic $\text{Log}_{\text{\setminus}} (\mathcal{C})$;
(2’) through representation show that $\text{Log}_{\text{\setminus}} (\mathcal{C}) = \text{MIL}_{\text{-Pre}} = \text{MIL}_{\text{-Pos}}$; and
(3) get decidability through FMP on $\mathcal{C}$. 

15
(Natural) extensions of $\text{MIL}_{\text{Pre}}$ and $\text{MIL}_{\text{Pos}}$ [and $\text{S4}$] are obtained by adding an informational implication ‘\’. 

**Definition**

The language is given by adding ‘\’ with semantics:

$$v \models \varphi \setminus \psi \quad \text{iff} \quad \forall u, w ([u \models \varphi, w = \sup\{u, v\}] \Rightarrow w \models \psi)$$

We denote the resulting logics as $\text{MIL}_{-\text{Pre}}, \text{MIL}_{-\text{Pos}}$, respectively.

The problems now become

(A\) axiomatizing $\text{MIL}_{-\text{Pre}}$ and $\text{MIL}_{-\text{Pos}}$; and

(D\) proving (un)decidability.

The same (1)-(2)-(3) structure is used as before, but now we

(1’) axiomatize the logic $\text{Log}_{\setminus}(C)$;

(2’) through representation show that $\text{Log}_{\setminus}(C) = \text{MIL}_{-\text{Pre}} = \text{MIL}_{-\text{Pos}}$; and

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(Natural) extensions of $MIL_{\text{Pre}}$ and $MIL_{\text{Pos}}$ [and $S4$] are obtained by adding an informational implication ‘\’.

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(Natural) extensions of \( MIL_{\text{Pre}} \) and \( MIL_{\text{Pos}} \) [and \( S4 \)] are obtained by adding an informational implication ‘\’.

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v \models \varphi \\text{\( \backslash \) } \psi \iff \forall u, w ([u \models \varphi, w = \sup\{u, v\}] \Rightarrow w \models \psi)
\]

We denote the resulting logics as \( MIL_{\backslash-\text{Pre}}, MIL_{\backslash-\text{Pos}} \), respectively.

The problems now become

(A\’) axiomatizing \( MIL_{\backslash-\text{Pre}} \) and \( MIL_{\backslash-\text{Pos}} \); and

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The same (1)-(2)-(3) structure is used as before, but now we

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(Natural) extensions of $\text{MIL}_{\text{Pre}}$ and $\text{MIL}_{\text{Pos}}$ [and $\textbf{S4}$] are obtained by adding an informational implication ‘\’.

**Definition**

The language is given by adding ‘\’ with semantics:

\[
v \models_\to \varphi \backslash \psi \iff \forall u, w([u \models \varphi, w = \sup\{u, v\}] \Rightarrow w \models \psi)
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We denote the resulting logics as $\text{MIL}_{\text{Pre}}, \text{MIL}_{\text{Pos}}$, respectively.

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(2’) through representation show that $\text{Log}_{\to}(C) = \text{MIL}_{\text{Pre}} = \text{MIL}_{\text{Pos}}$; and
(3) get decidability through FMP on $C$. 

15
**Question:** What happens if we extend $\mathbf{S}4$ with vocabulary for \textit{minimal} instead of \textit{least} upper bounds?

**Answer:** Nothing. We get the exact same logics:

$$\text{MIL}_{\text{Pre}} = \text{MIL}_{\text{Pos}} = \text{MIL}^{\text{Min}}_{\text{Pre}} = \text{MIL}^{\text{Min}}_{\text{Pos}}$$

and even

$$\text{MIL}_{\text{-Pre}} = \text{MIL}_{\text{-Pos}} = \text{MIL}^{\text{Min}}_{\text{-Pre}} = \text{MIL}^{\text{Min}}_{\text{-Pos}}$$

This concludes and summarizes our study of MILs on preorders and posets.
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This concludes and summarizes our study of MILs on preorders and posets.
Axiomatizing $MIL_{Sem}$

Three ways to completeness (some intuitions for our proof):

- **Henkin (e.g., $K$)**
  
  \[
  M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_\omega
  \]

- **Standard step-by-step (e.g., $MIL_{Pre}$)**
  
  \[
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  \]

- **‘Indeterministic step-by-step’ ($MIL_{Sem}$)**
  
  Model constr.:
  
  \[
  M_0
  \]

  \[
  \pi_0 \quad \pi_1 \quad \pi_2 \quad \cdots
  \]

  Axioms:
Axiomatizing $MIL_{Sem}$

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### Henkin (e.g., $K$)

- $M$

### Standard step-by-step (e.g., $MIL_{Pre}$)

- $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_\omega$

### "Indeterministic step-by-step" ($MIL_{Sem}$)

**Model constr.:**

- $M_0$
- $\cdots$
- $M_{01}$
- $\cdots$
- $M_{01n01}$
- $\cdots$
- $M_{0n0}$
- $\cdots$
- $M_{0n0n0n0}$

**Axioms:**

- $\pi_0$
- $\pi_1$
- $\pi_2$
- $\cdots$
Axiomatizing $\text{MIL}_{\text{Sem}}$

Three ways to completeness (some intuitions for our proof):

**Henkin (e.g., K)**

\[ \mathcal{M} \]

**Standard step-by-step (e.g., MIL$_{\text{Pre}}$)**

\[ \mathcal{M}_0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \cdots \rightarrow \mathcal{M}_\omega \]

‘Indeterministic step-by-step’ ($\text{MIL}_{\text{Sem}}$)

Model constr.:

\[ \mathcal{M}_0 \rightarrow \mathcal{M}_{01} \rightarrow \mathcal{M}_{011} \rightarrow \cdots \]

\[ \mathcal{M}_{0n_0} \rightarrow \mathcal{M}_{0n_01} \rightarrow \mathcal{M}_{0n_0n_01} \rightarrow \cdots \]

Axioms:

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‘Indeterministic step-by-step’ ($\text{MIL}_{\text{Sem}}$)

Model constr.:

Axioms: $\pi_0, \pi_1, \pi_2, \ldots$
Conclusion and future work

What we have done:

- Thoroughly surveyed the landscape of MILs on preorders and posets.
- Made crossings with the Lambek Calculus and truthmaker logics.\(^1\)
- Axiomatized $MIL_{Sem}$.

What comes next:

- Proving (un)decidability of $MIL_{Sem}$ and solving the ancillary problems of fin. ax. and the FMP w.r.t. $C_{Sem}$.
- Applying the techniques and heuristics of this thesis in other settings—not least those going into axiomatizing $MIL_{Sem}$.
- Further exploring how MILs connect to other logics.

\(^1\)See the thesis for this, including proofs of decidability (and compactness) of a family of truthmaker logics.
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• Applying the techniques and heuristics of this thesis in other settings—not least those going into axiomatizing $\text{MIL}_{\text{Sem}}$.
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Example
Note how `<sup>` and `\` are ‘inverses’:

\[ (\text{sup})p(p\to q) \to q \]

and

\[ p \to q\backslash(\text{sup}pq) \]

are valid.